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PROGRESS REPORT.

Mr. M. K. Kewalramani, B.A.—M.A. student, Fergusson College, Poona, has been elected a member of our Society at concessional rate.

2. Our Treasurer, Rev. C. Pollard, is proceeding to England on leave, and the Committee therefore thought it advisable to appoint Mr. S. Narayana Aiyar, M.A., F. S. S., Manager, Port Trust Office, Madras, to take up the duties of the Treasurer. The attention of the General Body is drawn to this change so that all matters pertaining to the Treasurer, may be referred to Mr. S. Narayana Aiyar during Mr. Pollard's absence.

3. *A Junior Trigonometry*—by Messrs. Borchardt and Perrott, has been received from Messrs. G. Bell & Sons, London.

The University of Illinois has presented to our Library the reprints of the Ph. D. Theses on the following subjects—

1. Invariants of Linear Differential Equations.
2. Projective Differential Geometry of Developable Surfaces.
3. Primitive Groups, with a determination of the Primitive Groups of Degree 20.
4. Functions of Three Real Variables.

Reprints of papers by Prof. G. A. Miller on Substitution Groups not exceeding the 7th degree, Groups generated by the two operators satisfying the condition $s_1s_2 = s^{-1}s^{n}_1$, Gauss's Lemma and some related Group Theory, Groups which contain an Abelian subgroup of prime index, on the use of Co-sets of a Group, Infinite Systems of Indivisible Groups, a Third Generalization of the Groups of the Regular Polyhedrons and some minor pamphlets have likewise been received.

POONA,
31st May 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

Mathematics in Relation to other Subjects of Knowledge.*

By Prof. R. Littlehailes, M.A.

On accepting the invitation to deliver your inaugural address my first feeling was one of cheery optimism, the subjects at my disposal being so many and varied that I felt certain of easily choosing one that would be acceptable to you all. On second thoughts however, numerous difficulties presented themselves. The subject of this inaugural address should be one that is fairly general in character, it should not be too technical, it should not be beyond the comprehension of even the least mathematically inclined of the audience and yet it should bear on the subject—mathematics—which offers the excuse for the existence of this Society.

I cogitated whether it would not be well to select the life of some renowned mathematician like Newton, Descartes, Euler, Lagrange or Laplace and give a sketch of his life and of the influence that he has exercised in the subject that we have so much interest in. Second thoughts, however, led me to choose the present title for the subject of my address this evening; in the first place because it is a relation that has always had a keen interest for me, in the second place because it is a subject that has reference to the *rationale* of the foundation of this Society, and thirdly because it is a subject that offers scope for exciting the attention not only of those who like ourselves take the subject of mathematics seriously, but also of those who by virtue of the modern necessities of their own particular special studies require its aid in their own work in a thousand and one ways.

There appears to me several ways of treating the subject of this address, but two of them stand out before the rest. I might narrate *seriatim* the various branches of human knowledge and ignorance and specify in detail the various branches of mathematics that are applicable to their complete study and research. On the other hand, I might catalogue the various branches and sub-divisions of mathematics—both *pure* and what is technically known as *applied*—and consider how each division or sub-division has been utilized by searchers after truth in other branches of knowledge; I have not adopted either of these methods to the exclusion of the other, but have compromised by accepting *in toto* neither the former nor the latter mode of exposition.

I have presumed, of course, that mathematics *has some* relation to other subjects of knowledge, the presumption appearing to me so obvi-

* An address delivered at the inaugural meeting of the Mathematical Association of the Presidency College, Madras, 1914.

ous as hardly to require notice. I might even go so far as to state that I am willing to show the applicability or relationship that exists or might exist between it and any other subject that might at random be named. Having that in mind it is all the more difficult for me to render this address in any way complete or systematically detailed. It must be and must remain—unless you were willing to sit for a course of lectures on this subject—of a cursory character ; more of the nature of a brief survey, than of an exhaustive treatise wherein logical sequence of unimpeachable reasoning is the main criterion of excellence.

And this brings me naturally to one of the first divisions of my subject—the reasoning powers of mathematicians and the more general question of the philosophy of reason : a subject that is no doubt often well debated by philosophers who know no mathematics, though the converse, *viz.*, the mathematician who knows no philosophy, scarcely exists ; for, who can be a mathematician without being at the same time a philosopher, a lover of wisdom, even if it be but a small portion of the wisdom of the universe ? One of the recent tendencies of modern mathematics is the intense study of first principles, a profound introspection and critical examination of the foundation principles of the subject; its fundamental concepts and methods have been subjected to most careful scrutiny and the part played by in tuition and by logic, with the necessary limitation to each, have received the most careful attention of several of our most renowned thinkers, in particular of Whitehead and Russell. Their written cogitations take us immediately into the realms of "Metaphysics"—the careful preserve of the philosopher. This subject has been variously defined, by both the more and the less seriously inclined ; its abstruseness is probably well represented by the cynical definition that the study of metaphysics is akin to "a blind man searching in a dark room for a black cat which is not there." And to quote merely one of its applications to mathematics, I might state that we involve ourselves in apparently contradictory but in reality absolutely comprehensive statements : any connection between a and b may be considered as a relationship between a and b ; if there be no connection between a and b, the statement that there is no connection is itself the statement of a relationship between them. This of course means that there is nothing in nature which does not bear some relationship to any or every other thing in nature,—a negative attribute being as essential, as a positive attribute, in the definition and limitation of a relationship. I do not however desire to dwell upon the relation of mathematics to philosophy and in particular upon that portion of the philosophy of mathematics which makes it self-evident

that mathematics is related to every other subject upon which concepts may be formed and thoughts uttered.

I shall turn my attention to what I consider, in these essentially practical, industrial and commercial modern times, one of the most important features of mathematics ; *viz.*, its relation to what is currently known as Science. Specific branches of science *e.g.*: the treatment of Dynamics in its extremely large number of branches are recognised subjects of study and research of students of mathematics and to detail every connection between pure mathematics and mechanics would be as tedious as the compilation of a Tamil Dictionary, though possibly much more interesting. I shall however have to touch upon one or two main connections shortly, but before doing so I desire to emphasize the attitude taken up by the more ardent investigators in the various fields of science ; it is that no real and true science can exist until the phenomena pertaining thereto can be and are measured quantitatively and not merely observed qualitatively. That a phenomenon exists and is observed is the first step towards the formulation of any science, but that is far from being the whole method of science. Observations should then be made of the general tendency of the phenomena, whether there are any changes in time, any changes in space, any changes in mass, and when these changes are observed the variations are classified and we reach such a science, as for instance, "systematic botany." There is however no true science until the aid of mathematics is sought and applied. That varieties, variations and changes exist is interesting and these divergencies offer food for a certain amount of more or less inexact thinking and reflection such as a child in its earlier stages of development may be expected to indulge in, but the mature rational mind, especially if that mind has received a logically scientific training, cannot rest content there. It must *measure* the *directions* of these variations and their *magnitudes* both with regard to space and with respect to time ; and it is only at this stage that the phenomena come to be classed as truly scientific and can be treated in a really scientific manner. No real advance can be made into the science of any phenomena until the aid of mathematics is sought.

Some sciences like the so-called natural sciences have come under the influence of mathematics to an extremely small degree, while others like physics are so interlocked with mathematics that the student of physics cannot proceed without calling in the aid of the mathematician at almost every step. Let us glance, for a moment, at one of the natural sciences, geology for example, and let us take one of its sub-

groups, mineralogy. In the treatment of this subject how is mathematics involved? To begin with there are the various types of crystal symmetry which are best discussed about three axes, and the student of analytical geometry of three dimensions looks upon symmetry in crystallography as a mere corollory to some of his more elementary mathematical ideas connected with three dimensional space. Further the determination of different kinds of minerals and in particular of crystals, is simplified by their behaviour with regard to light, and especially as regards their behaviour in polarised light by means of which some of the most beautifully coloured pictures of symmetry and of geometric shapes are evidenced. But how are we to further investigate the formation of these shapes and colours? They are investigated by a consideration of the geometry of curves and surfaces coupled with some theory of light. And how can we investigate or even postulate any theory of light without mathematical analysis at our elbow? Is there any student of physics among you, who can tell me that he has discussed the wave theory of light, refraction, polarisation, wave lengths, interference or any of the current ideas and phenomena of optical science without calling to his aid mathematics? And have not many of you felt that this particular mode of treatment of the subject, or that special little piece of analysis, was difficult, only because it was not completely comprehensible without a prior mathematical training? It is probably too well known among you that the fixation of geological dates is another geological subject that has given rise to mathematical calculations based upon cooling phenomena, and it is needless for me to spend more time on that subject than that taken in mentioning it; Kelvin's writings on the secular cooling of the earth show to geologists and physicists alike the use and application of mathematics to their respective branches of knowledge. I might choose other sub-divisions of geology and explain how they have been shown or might be shown to depend for their comprehensive treatment on a knowledge and application of mathematics; but I must pass on to some other subject.

In physiology we do not rest content until we have measured the *amount* of change in the circulating system due to respiration, the *time* of reflex action and until we have taken other more or less absolute measures. These merely form the basis for other estimates and determinations which have as one of their ultimate objects the meaning of life and death from the physiological standpoint. It would be of extreme interest to show the relationship that exists between mathematics and physiology, but I am not qualified to do more than mention that one of the

modern tendencies therein is, as it is in all other subjects, to demand mathematical exactness in all observations, calculations and inferences.

The science of medicine is more nearly allied to that of chemistry, chemical physiology or physiological chemistry being one of the most important branches of modern science, and so we now turn our attention for a moment to the interesting subject of chemistry which assumes a discussion of the atom as pre-eminently its own. We shall not quarrel with her on that ground. We can well afford to be magnanimous. We know that chemistry without the atom is almost as insipid as rice without curry and so we freely give the atom to the chemist to play with. But he cannot play very long with his atom without experiencing an anxious desire to know something more about it than that it is his fundamental postulate, and he turns to the physicist to aid him in a discussion of its behaviour under certain circumstances and particularly under the action of heat. The physicist is delighted at the honour paid to him (though he considers in his innermost self that it was obvious to even the meanest intellect that his aid was all the while necessary) and sets forth to propound his ideas of the atom and its behaviour. But even he has not gone far before he finds himself in almost as complete a *cul de sac* as the chemist was before him. He finds that he is confronted with certain phenomena which can be very well expressed by means of mathematical equations ; but the discussion of these equations, their complete solution and re-interpretation in terms of the original physical concepts are problems that he does not feel quite at home with. He has no other resource than to call in his friend the mathematician who lays the store of his analytical wealth at his feet. Let us make the relationship of chemistry to mathematics a little more concrete by abstracting from any chemical syllabus. I quote from the B.A. syllabus of the University of Madras. "The law of mass-action ; the velocity of chemical change ; the relations of chemical energy to heat, and to electrical energy,......the kinetic theory of gases." It does not need a specialist to see that in these portions of chemistry the chemist seeks the aid of the physicist and of the mathematician. Even now some on you will have recollections of the haunting fear that is inspired in you at the recollection that thermodynamical formulæ are equations that your lecturer and professor expect you to have heard of, to have seen and to have read about ; nay, more, your professor possibly expects you to be acquainted with them ; he may even expect you to be so enamoures of them as to spend some odd minutes—hastily snatched from watching an interesting cricket or football match or from participating in a game of tennis—in endeavouring to solve problems on the thermodynamics of

a gas that is not perfect. An awful nightmare it must be to the non-mathematical chemist to have such a vision called up before him.

But the chemist is not alone. The physicist in the advanced stages of his researches grieves at the amount of mathematics that he is required to be acquainted with. He is even required to be at home several afternoons each week to "the calculus" and his friends. Indeed he has come to recognize that he is completely out of the social "world of accurate science" unless he has frequent visits from the universal friend, "mathematical analysis." How could he consider problems on the conduction of heat without "Fourier's Series," and how can Fourier's series be considered if not in connection with general analysis?

I might now turn to one branch of physics that is of so universal an application that we can only just touch upon its fringe. I mean electricity with her sister magnetism. Electricity is such an immense subject that I find some difficulty in knowing what particular branch should be first instanced as leading up to or depending upon mathematics for its investigation. I might instance one of the smallest and yet most important equations that it is the pleasure of the electrician to deal with, I mean the equation of motion of a magnet situated within a coil around which a current momentarily flows and which is represented in various forms but always as a differential equation of the second order. It is indeed the same equation as we meet with in the study of oscillatory motion in rigid dynamics and whose solution denotes either periodic or aperiodic motion. The mathematical study of this equation in its various forms is not difficult and its application to the throw of a galvanometer, to the swing of a pendulum and to many other natural phenomena with which you may or may not be acquainted is extremely interesting, first of all in the application of the equation to the motion under discussion and secondly—and this is perhaps of even more importance—in the interpretation of the solution of the equation in terms of the physical constants from which the original differential equation was formed.

Again, electromagnetic theories open up a large field of mathematical research and are so closely allied to radiation theories—either as offering material for analogy, or as giving opportunity for controversy—that it is no wonder that the advice of the mathematician is sought to reduce the theories to symbols, to criticise the deductions comparing the true with the doubtful, and to state carefully and analytically the suppositions upon which any hypothesis is based and the conclusions to which any accepted hypothesis must lead to. It would be beyond the scope of a general talk of the present nature

to enter at any length upon a discussion of the theories of radiation and electromagnetism; it remains to emphasize the fact that theories based upon experiment and propounded by physicists must be subjected to the critical examination of the mathematical analyst just as the theories that are propounded by the mathematician have to be subjected to the vital tests of conformance with experimental phenomena such as are observed by the physicist.

It is hard to leave the subject of physics in such a discussion as we have before us at present. We could continue with a reference to the theory of sound, its vibrations, its transmission of energy and all its mathematical representations which have been so ably exposed by such masters of the subject as Helmholtz and Rayleigh, both of them mathematicians of no mean order. We could refer to the mathematical theories of the conduction of heat, to the mathematical discussion of elasticity with its accompanying stress and strain about which volumes of mathematics have been written. We could touch upon probably the most interesting of modern mathematical researches—those which bear upon air-ships and promise to revolutinize some of our present means of travel. Aero-dynamics is a fascinating subject from the standpoint of the man in the street; but, the mathematician recognises that the aeronaut desires that his life shall be in no greater danger than that of the motorist and he wishes to trust himself only to such an aeroplane as is proved by mathematical analysis as well as by experiment to be one in which the stability of its motion is assured under even the most adverse conditions. The application of such analysis is only now being made; the theory is, in places, heavy and based upon various suppositions of fluid resistance and strength of materials, but advances are foretold and the aeroplane of the future will without doubt be modelled from data that have mathematics as the guarantee for its stability of motion.

Hydraulics and fluid resistance are met with in the construction of ships and also in a discussion of one of the modern means of developing energy, namely, in the turbine and in particular the steam turbine, while a knowledge of the strength of the materials used in the construction of all machines is necessary; this again reverts to a consideration, among other things, of the elasticity of the substance used in construction. We must not however, at this stage, claim too much for the subject of our heart. Mathematics helps a very great deal in the development of all modern sciences, yet in the commercial and industrial life of the scientific world a great deal of empiricism forms the basis of its constructions. A certain thickness of an iron bar is used

for a certain portion of a machine or building, not because it has been proved mathematically that this thickness is necessary to withstand the strain that is likely to be produced by the stress, but because experience and experiment have proved that this thickness gives a suitable quantity of material able to bear the necessary tension or thrust. Mathematics enters here only by gradually systematising the forces at work upon the several portions of a machine and of reducing these forces with their actions and interactions to their fundamental constituents and indicating the necessary strength of the several parts that is required to withstand the applied forces. And in this there remains an immense amount still to be done by engineering mathematicians and mathematical engineers. One could wish that a larger number of such existed ; for it is deplorable to find so much empiricism and acceptance of dubitable authority on the part of engineers due to a great extent to the magnitude of their subject and in part, no doubt, to the disfavour with which many of the most capable mathematicians that the world has produced, look upon the apparent degradation of their higher mental outlook when applied to the ordinary commercial and industrial facts that they so often scorn. It is further to be remembered that a very large number of the applied mathematical problems of to-day did not exist a century ago. With the advent of steam and oil engines and of the application and use of electrical forces, new problems have arisen which are increasing day by day and are of necessity first attacked experimentally as problems of experimental or industrial science before the aid of the mathematician is sought.

It is to be regretted that the mathematical treatment of a subject is in general only sought when it is found that it can make little further advance upon purely empirical and experimental lines. Who, for example, sought to discuss with any fullness the viscosity and lubricating powers of various oils and to what extent variations in surface tension were dependent upon differences of temperature until it became necessary to lubricate fast moving machinery—a product of our more modern industrial civilization ? In these discussions mathematical equations whether they are the simple ones of an ordinary readily soluble linear algebraic relation or the somewhat more difficult differential equations arising in oscillatory motion, or in thermodynamics or other relations, and which may not yet be completely solved by the ordinary analysis at our disposal,—are invariably used to represent the physical phenomena observed. Their ultimate solution is necessary and, to my mind, certain.

But let us turn aside for a while to the application of mathematics to subjects of study that do not at first sight appear to allow of any applicability. We may begin with history and economics—two subjects which are usually coupled together though there is no obvious necessity therefor. Mathematics, physics, geology, in fact, any subject of knowledge can be treated historically—just as economics so frequently is. It is to certain aspects of economics that I would invite your attention for a moment. The principles of economics are now being investigated from the standpoint of statistics and the treatment of statistics is essentially a mathematical matter of some importance. In it we are lead to a study of frequency curves and of correlation co-efficients. Modern statistical methods are based upon the theory of probability and are usually associated with the name of Professor Karl Pearson. I have hopes that the University of Madras will see its way, during the next academic year, to invite some statistical authority to Madras for the purpose of delivering a course of lectures of both an elementary and an advanced character on statistical methods. I cannot however do better here than give the ordinary definition of statistics as the science of averages. It would please me to enter into a discussion of averages, modes, means, frequency curves, correlation and probable errors and of their application to the study of economic factors but it will suffice if I refer you to the papers of men such as Pearson, Edgeworth, Elderton, Yule and Bowley. Statistics is a branch of applied mathematics that has unfortunately not received as much attention as it deserves in this country where there are vast accumulations of data awaiting analysis by some one with the necessary mathematical training and with much time at his disposal. Some future research scholars have herein a suggestion for their subject of investigation. "The method of least squares" which used to find so much favour with scientists and indeed at the present day does still find a large amount of adherance on their part is only one method of mathematically treating a series of recorded observations, and it can give place to their treatment by such frequency curves as appear best fitted to the range of observations that may be under discussion.

Questions based on the theory of probability naturally lead us from statistical economics, as distinguished from historical economics, to a matter that we all think of sooner or later, the probability of the length of our life and the precautions that we all should take to ensure that those dependent upon us may at our death not be left unprovided for. The Hindu Joint Family system does not necessitate life insurance and assurance on the part of Hindus to such an extent as it does on the

part of members of other Nationalities, but with the breaking up of the joint family system and the dispersion of the units that form the family, the necessity for such a provision urges itself forward. It is not every one who thinks how it is that one insurance company charges larger premiums than another for the insurance of a life, or how the sum fixed as a premium is arrived at. Considerations both theoretical and experimental govern the selection and treatment of such statistics as form the basis of the various tables that are used in fixing the sums required to be paid as premiums—tables of mortality, of sickness, of succession, of marriage, of superannuation and so forth. All these are to be used by the actuary and must be prepared by him or by his friend the mathematical statistician. Actuarial science may be defined as the collection and utilization of statistics for the purpose of calculating life contingencies—a narrow definition; in a wider sense it is the collection and analysis of past experience and the employment of the results of such analysis to forecast the average future. This analysis may be explained in simple mathematical language as the formation of certain curves which trace out the relationships between the real varibles concerned as deduced for a very large number of observations (*e.g.*, between the age of a person and the number of persons living at that age) and the application of these average curves to any particular case in point. This treatment is essentially mathematical, though the standard of mathematical knowledge required by the student is no exceptionally high and I suggest that some of you might in the near future apply your mathematical knowledge in the direction of statistical analysis.

The hypercritical listener may at this stage ask how mathematics is allied to the study of certain other branches of knowledge, for instance, the study of English and of languages in general. I would not pretend to state other than that the connection is practically nil. We here come to the main distinction between what is known to the educationist as the difference between literary studies and scientific studies. The student of literature may go through the world with his reasoning faculties trained to a very slight extent, the student of science, however cannot make any real satisfactory progress in his work unless his reasoning faculties are developed and one of the surest and safest methods of developing these faculties is by means of an accurate mathematical training. I have long known that the Hindu has many excellent characteristics but that it is necessary for these characteristics to be qualified by a course of mathematical training, so that the average Hindu mind may be reduced from what may be mathematically termed as a tendency to

state of small oscillation about a position of relatively unstable equilibrium to a tendency to a state of steady motion about a position of mental equilibrium. This training may be given by a study of other subjects than mathematics, but the value of these subjects is dependent upon the amount of reasoning that is required from the student, and without doubt this training of the reason is more readily found in a study of mathematics than in any other subject. We frequently hear it stated that the Hindu lacks "common sense." It has been my good fortune to find this remark applicable to extremely few of the Indians with whom I have come into close contact; it may be that the remark is one that has very limited reference, but even if there is the smallest modicum of truth in it, I am convinced that my experience is the result of the fact that "common sense" has been developed by the rational training that every successful student of mathematics is compelled to undergo.

I desire now to touch upon the "non-humanising" influence that a study of mathematics is supposed to have upon all its ardent students. I have heard the same statement made about students of other subjects, *e.g.*, about the student of metaphysical philosophy who enjoys such literature as Kant's *Critique of Pure Reason*. I confess that there appears to be a little, but very little, truth in the idea. Mathematics is essentially the subject of the specialist and every specialist is presumed to be slightly un-human, but I comfort myself with the thought that there is something more than human and the non-humanising influence of mathematics is the development of the human mind beyond that which is human into that which is more than human. On the other hand, if we pick up any book on the "History of Mathematics" and read of the development of mathematics from the earliest recorded times, if we glance into the actual lives of the men who are only known to many of us as mere names, we will find that they lived, ate, drank, married, begot children and died just as we do now-a-days; we will read that they had their worries and trials, their poverty and honour and that they eventually went the way of all flesh, forgotten by the majority of the world and occasionally remembered only by a student as a mere name; as for instance in Newton's Law of motion, in Bessel's function, or in Cartesian co-ordinates. We then experience the feeling that the mathematician cannot be singled out to stand at the corner of the street and thank his God as the Pharisee did, that he is not as other men are.

One fact I should like to impress upon you at this stage; very few of the older students appreciate it and practically none of the younger students know it. It is that mathematics like all other sciences is a

living science. The development of this idea would occupy too much time and I would only just mention one way in which this is brought home to us. The inter-dependence of pure and applied mathematics, as for instance in the application of mathematics to questions arising out of the study of natural phenomena, frequently produces a problem which is mathematically insoluble by the methods of pure mathematics now at our disposal. This necessitates therefore further investigations into and developments of pure mathematics, and in particular of mathematical analysis; thus the progress of mathematical science is emphasised and history is made.

Another branch of science to which I might invite your attention as a subject of investigation is meteorology. Those of you who have been sufficiently interested in this fascinating subject will have recognized that until very recently practically all our observations have been made in two dimensional space. Much can be made and indeed has been made of an analysis of meteorological data of two dimensions but there is apparently no hope of completeness or finality in our deductions until we can measure pressures, temperatures and the like, simultaneously at places *above* the surface of the earth, as well as at places *on* its surface. This, you might retort, is the work of the physicist. I agree: but as soon as the observations are made and collected, it behoves the mathematician to step in and proceed with his analysis of the amassed figures. Meantime, many of the figures collected for stations in India on the surface of the earth await investigation, and I throw out the suggestion for your application.

But I have occupied your time too long already and must bring these rather scattered remarks to a speedy termination. It would ill become me, however, to close an address of this kind and in this country without some reference to astronomy, the study of which dates back almost as far as the origin of history. To enlarge however on the connection between astronomy and mathematics to this audience would be superfluous and I can only ask you to bear astronomy in your mind as another subject in which the mathematician is absolutely necessary.

Finally, I thank you for the honour you have done me in requesting me to deliver your inaugural address and I conclude wishing your Society every good wish for its future prosperity.

Efficiency in Elementary Mathematics.*

To-day the dominant note in education is efficiency. It is fitting therefore that we should consider the factors for training in efficiency in elementary mathematics.

The expressions of opinions, serious comment, great thoughts, inventions, social and intellectual achievements are not spontaneous, nor do they occur in a single mind because they are results of race development. How often in our experience have we been spared from expostulating on some cherished idea, design, method or invention because accident has given priority to some other independent worker. Such first announcements are apt to be vague and undefined and as a consequence men take issue and often give way to passionate disregard and intolerance for the tenacious attitude of an adversary.

I have a pretty firm belief in that greatest of paradoxes, *viz.*, that both sides of an argument are right. Therefore, my friends, I am going to regard your assertions relating to the cultural and utilitarian value of any mathematics material as amphibolous aspects of an ideal future vocation. I am not going to reconcile these aspects, because they need no reconciliation. If you will permit me to state an analogy, it appears to me as if some of us had gone into a long glass tube, while others remained outside. In so doing we have chosen our limitations of vision. When we recognize the fragile barrier that separates our groups we can understand why we look upon one another telescopically or microscopically. We owe to a heritage the fallible belief in a dualism which has been symbolized constantly in the development of our literature, art, religion and science. The sense of opposites expressed as north and south, hot and cold, good and evil suggest culture and utility. Let us give praise to that noble soul who first enunciated the law of *relativity*. Now you know there is no greater avenue of adventure than higher geometry which reveals the infinite attenuation of life and comprehends the processes which guide the unborne possibilities of the future.

The isolation of the subjects of the old curricula has produced sterility.

"Education has not been able to keep abreast with the innumerable problems of our complex life very largely, because its emperical methods are obsolete."

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* Extracted from a paper read by Mr. E. H. Koch Jr., of New York before a Mathematical Association.

"Even scientific men are recognizing the fact that they must apply scientific methods to determine the circumstances which promote or hinder the advancement of science."*

It is surely time for educators to apply the scientific method to determine those factors which enter into the problem of the efficiency of educational systems and more particularly to the special problem regarding the training for efficiency in elementary mathematics.

The best definition of efficiency is borrowed from the technical man who first ascribed it in a commercial way to a machine. Efficiency is the ratio of net output to gross input.

$$(a) \qquad \text{Efficiency} = \frac{\text{output}}{\text{input}}.$$

The human being is also a machine in the sense that it is a highly organized complexity of more or less co-ordinated parts. The intellectual attributes when associated with the nervous system bear to the framework of the body the analogous relation that the forces of nature bear to structural material.

Efficiency is a measure and is therefore expressed as a ratio between a realized performance and an ideal possibility. There are as many different aspects of efficiency as there are considerations in any mental or physical situation. When you speak of health or strength you unconsciously give efficiency a special name.

It was quite natural to expect technical men to apply this term to the situation which presents itself in technical and industrial education, because the facts in their case were more definitely known. In this respect they have made a beginning.

It is my purpose today to indicate how such an application can be made in the special field of mathematics. Let us consider the summary of our technical colleagues who volunteer the following meanings of efficiency applied to education* :—

1. The efficiency of the curriculum—the ratio of the useful information and mental training offered by a college to that which an ideal college should offer.

$$\text{(I)} \quad E_c = \text{Efficiency of curriculum} = \frac{\text{what is offered}}{\text{what should be offered}},$$

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* Science November 4th 1910.

* Karapetoff: *Contributions to the Discussion of Efficiency in Engineering Education*, S.P.E.E. Bulletin, Vol. XVIII.

2. The efficiency of the teacher—the ratio of the amount of knowledge and mental development acquired by the student to the amount of the opportunities offered by the college.

(2) E_t=Efficiency of the teacher

$$=\frac{\text{what and how he teaches}}{\text{what and how he should teach}},$$

3. Efficiency of the student as receiver—the ratio of that which the student assimilates to that which college offers.

(3) E_r=Efficiency of student as receiver

$$=\frac{\text{what he assimilates}}{\text{what the college offers}},$$

4. Efficiency of the student as giver—the ratio of the use made by the student of his college education during his professional career, to the potential energy stored in him at the time of graduation.

(4) E_g=Efficiency of student as giver

$$=\frac{\text{what he gives to the country}}{\text{what he has received from the college}}.$$

There was a time when the soldier was the most efficient man in the State. Centuries later the mantle of efficiency befell to the lot of the lawyer. The evolution of the political, social and industrial progress in the development of civilization compels the finger of time to point to the men of science as the dominant leaders of the nations.

"In America at least", says Professor Townsend,* "we have come to accept as a fundamental principle that the supreme test of an education is the efficiency of the training it gives the individual to meet the demands of organized society and at the same time enable him to contribute most directly or indirectly to the general progress of national life."

Therefore

(5) E_m=Man's overall efficiency

$$=\frac{\text{his productiveness}+\text{his contributions to society}}{\text{latent possibilities times education in its larger sense}}.$$

(6) $$E_m=\frac{P+C}{LS},$$

where P=productiveness,

C=contributions to society,

L=Latent possibilities,

S=education in its larger sense.

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* Science, November 4, 1910.

Each one of these quantities is resolvable into innumerable factors many of which although not definitely known are nevertheless familiar bywords in the parlance of educational meetings. The unknown factors are equally important and must be determined and investigated. The task at first thought appears so stupendous as to seem almost insuperable, but if we apply the scientific spirit we need not despair. Some years ago the Government sought to devise an automatic machine for predicting the tides for any locality, and for long stated intervals in advance, The question was carefully undertaken, although the obstacles seemed as insurmountable as the elements entering into weather prognostications. The working equation contained thirty-five variables. Today that machine is a realization. Our problem narrows down when we confine our attention to the elements of mathematics which contribute to the efficiency of the individual.

The individual may be likened to the incandescent lamp which has an efficiency of approximately one per cent. This means a waste of ninety-nine per cent. of the energy stored in the coal. Briefly told the final efficiency is the product of all the contributing efficiencies involved in the mechanical and electrical transformations and transmissions through boiler, engine, generator, wires and filaments of the lamp.

(A) E_s = Efficiency of the system

$$= \frac{\text{energy given out as light.}}{\text{energy stored in the coal}}$$

$$\text{(B)} \quad E_s = \frac{\text{Energy stored in the coal} - \text{total losses of energy}}{\text{energy stored in the coal}}.$$

In like manner, man's overall efficiency may be expressed in terms of losses.

$$(8) \qquad E_m = \frac{LS - (O + I + F)}{LS}$$

substituting in (6) and (B).

O = lost opportunities,

I = Impediments,

F = Failures.

$$(9) \quad \therefore \qquad P + C = LS - \{O + I + F\},$$

substituting in (8) and (6).

Or we can represent man's overall efficiency as the product of all the efficiencies in the educational power station.

(10) $E_m = E_c \, E_t \, E_r \, E_g$. See (1) (2) (3) (4).

We are especially interested in the mathematics applied to Ec. We shall define 'mathematics efficiency' as the ratio of the factors which

contribute to man's social, political and industrial abilities to the stimulus and power given by mathematics training.

$$(11)\qquad E_m = \text{Mathematics efficiency} = \frac{\text{contributing factors}}{\text{stimulus and power given}}.$$

$$(12)\qquad E_m = \frac{X}{M}.$$

$$(13)\qquad X = f\,(x_1x_2x_3x_4\ldots\ldots x_n),$$

$$(14)\qquad M = \phi\,(m_1m_2m_3\ldots\ldots m_k),$$

$$(15)\quad \therefore \qquad E_m = \frac{f(x_1x_2x_3\ldots\ldots x_n)}{\phi(m_1m_2m_3\ldots\ldots m_k)}.$$

$x_1x_2x_3\ldots\ldots x_n$ are the factors affected by mathematics which contribute to man's social, political and industrial abilities, and X is a function of $x_1x_2\ldots\ldots x_n$;

$m_1m_2m_3\ldots\ldots m_4$ are the potential contributions of algebra, geometry, analysis and mechanics, and M is a function of $m_1m_2\ldots\ldots m_k$.

There are two ways of increasing E_m, either increase the numerator X, or decrease the denominator M.

The known *m's* are usually summarized thus :—

m_1 = the notions of number, measure, form, element,

m_2 = the focussing of law through symbols and graphics,

m_3 = the visualization and interpretation of law through formulas and graphic representation,

m_4 = the appreciation of co-existences and sequences of phenomena,

m_5 = the projection of the intellect beyond experience,

m_k = not defined.

We shall next turn our attention to the *x's* and challenge their validity. We have been talking about these *x's* for a long time and dare say have invented a few out of mere deference for an educational hobby. We must not forget one of the staunchest and earliest critics, Profesor Perry* who holds that the study of mathematics began because it was useful, it continues because it is useful, and it is valuable to the world because of the usefulness of its results in

x_1 = producing the higher emotions and giving mental pleasure,

x_2 = brain development,

x_3 = producing logical ways of thinking,

x_4 = aid given by mathematical weapons in the study of physical science.

* British Association Meeting, Glasgow, 1901.

x_5=passing examinations,

x_6=giving men mental tools as easy to use as their legs or arms, enabling them to go on with their education,

x_7=teaching a man the importance of thinking things out for himself and so delivering him from the present dreadful yoke of authority,

x_8=making men feel that they know the principles,

x_9=giving acute philosophical minds a logical counsel of perfection altogether charming and satisfying.

"Any subject may be useful as applicable to some special purpose or need of life or it may be useful as affording valuable mental discipline."

The following x's were excerpted from various sources :—

x_{10}=mathematics should promote culture, which means the subjugation of imitative power to the creative so as to make the individual develop centrifugal force, individuality, critical opinion and transform that which is read into conversation and life,†

x_{11}=mathematics should maintain the standards of a liberal education which implies a sense of truth and beauty,

x_{12}=mathematics should promote the identical spirit of science and literature which is to transform something of value from the unknown into the realm of the known,

x_{13}=mathematics should respond to the dominant activities of the nation,

x_{14}=mathematics must justify itself in the acquisition of information of commercial value,

x_{15}=mathematics should provoke intellectual tolerance—give breadth of thought,

x_{16}=mathematics should stimulate enthusiasm.

x_{17}=mathematics should instil courage and sympathy with life,

x_{18}=mathematics should give a realization of the value of human life as a public asset,

x_{19}=mathematics should promote intelligent leadership in matters of public concern,

x_{20}=mathematics should promote economy of time and effort,

x_{21}=mathematics should promote deligence and consciousness.

x_{22}=mathematics should promote strength of character,

x_{23}=mathematics should promote judgment,

† Professor Aston, Columbia University, September 28, 1910.

x_{24} = mathematics should promote fidelity and poise,

x_{25} = mathematics should pronote neatness and accuracy,

x_{26} = mathematics should train in habits of attention,

x_{27} = mathematics should train in system in dealing with material,

x_{28} = mathematics should train in clear and interesting presentation both oral and written,

x_{29} = mathematics should render service in the development of the national resources, in aiding the growth and expansion of its industries and of its commercial power and in the conservation of the resources that constitute the inherited wealth of a people,‡

x_{30} = mathematics should furnish a solid foundation on which the more advanced work of college and technical school may be based,

x_{31} = mathematics should develop an appreciation of the methods and spirit of pure science,

x_{32} = mathematics should lead to precision,

x_{n} = last but not least, mathematics should promote common sense,

There are many more factors of the numerator which may be mentioned. The critical test must be applied to each x, the so called desirable accomplishment, to see if it rings true or whether it has been coined to defend in a vague way some pet fancy of the mathematics department. It is not generally true that in determining the efficiency of any branch of study these same accomplishments are more or less in advocacy. It seems to me that if this were true we would be wasting effort and time in accomplishing in an inefficient way what may be more efficiently accomplished in another subject.

Beyond the ordinary claims mentioned which have been advanced and reiterated is the potent claim of a very broad consideration. I am convinced that in every department of human investigation there are definite underlying mathematical relations which are either little understood or totally unknown.

‡ See Science and Public Service Address, University of Illinois Biologic Station, July 22, 1910. "American Educational Defects," Science, October 28, 1910. Report of Training of Mathematical Instructors, Bulletin A. Math. Soc., November, 1910.

SHORT NOTES.

On the Integral $\int_0^\infty \frac{\cos rx dx}{(1+x^2)^n}$ and some allied Integrals.

§ 1. The solution of Question 472* by Prof. K. J. Sanjana admits of further generalisation as follows :—

Let I denote $\int_0^\infty \frac{\cos rx}{(1+x^2)^n} dx$; differentiating $2n$ times with regard to r, we get

$$\frac{d^{2n} I}{dr^{2n}} = (-1)^n \int_0^\infty \frac{x^{2n} \cos rx}{(1+x^2)^n} dx ;$$

or,
$$(-1)^n . \frac{d^{2n} I}{dr^{2n}} = \int_0^\infty \frac{x^{2n} \cos rx}{(1+x^2)^n} dx.$$

Similarly

$$(-1)^{n-1} \frac{d^{2n-2} I}{dr^{2n-2}} = \int_0^\infty \frac{x^{2n-2} \cos rx}{(1+x^2)^n} dx ; \text{ etc.}$$

Proceeding in this way, we shall get

$$(-1)^n \left\{ \frac{d^{2n} I}{dr^{2n}} - n_1 \frac{d^{2n-2} I}{dr^{2n-2}} + \frac{n_2}{2!} \frac{d^{2n-4} I}{dr^{2n-4}} - \dots\dots\dots + (-1)^n I \right\}$$

$$= \int_0^\infty \frac{\cos rx}{(1+x^2)^n} \left\{ x^{2n} + n_1 x^{2n-2} + \frac{n_2}{2!} x^{2n-4} + \dots\dots\dots 1 \right\} dx$$

$$= \int_0^\infty \frac{\cos rx}{(1+x^2)^n} \left\{ 1+x^2 \right\}^n dx$$

$$= \int_0^\infty \cos rx dx = I_0, \text{ (say).}$$

But I_0 is known to vanish (Todhunter, § 291) ; hence we have the differential equation

$$(-1)^n (D^2-1)^n I = 0,$$

to determine I_n, where $D \equiv \frac{d}{dr}$.

Solving this, we get

$I = e^r (A_0 + A_1 r + A_2 r^2 + \dots A^{n-1} r^{n-1}) + e^{-r} (B_0 + B_1 r + \dots B_{n-1} r^{n-1})$.

* Journal I. M. S., December 1913, pp. 230-31.

Now I is finite throughout the range of integration and remains finite however large r may be; hence the terms in e^r must be absent, or $A_0, A_1, A_2 \ldots\ldots A_{n-1}$.
must be each absolutely zero. Thus

$$I = e^{-r}(B_0 + B_1 r + B_2 r^2 + \ldots\ldots B_{n-1} r^{n-1})$$

and we have now to determine B_0, B_1, B_2, etc.

As $e^r I = B_0 + B_1 r + B_2 r^2 + \ldots\ldots + B_{n-1} r^{n-1}$, we obtain, differentiating both sides k times,

$$e^r \{ I + kI' + \frac{k_2}{2!} I'' + \ldots\ldots k \cdot I^{(k-1)} + I^{(k)} \}$$

$$= \underline{|k}\, B_k + (k+1)k \ldots\ldots 2 \cdot B_{k-1} r + \ldots (A)$$

where the accents denote differentiation with respect to r.

To get B_k we put $r=0$ on both sides.

Observating that $I = \int_0^\infty \frac{\cos r x}{(1+x^2)^n} dx$, we get when $r=0$,

$$(I)_0 = \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

Again $$I' = \int_0^\infty \frac{-x \sin r x}{(1+x^2)^n} dx, \text{ so that } (I')_0 = 0;$$

similarly $$(I''')_0 = (I''''')_0 = \ldots\ldots = 0.$$

Again $$I'' = -\int_0^\infty \frac{x^2 \cos rx\, dx}{(1+x^2)^n},$$

$$\therefore \quad (I'')_0 = -\int_0^\infty \frac{x^2\, dx}{(1+x^2)^n}; \text{ etc.}$$

We thus get from (A),

$$\underline{|k}\, B_k = \int_0^\infty \frac{dx}{(1+x^2)^n} - \frac{k_2}{2!} \int_0^\infty \frac{x^2 dx}{(1+x^2)^n} + \frac{k_4}{4!} \int_0^\infty \frac{x^4 dx}{(1+x^2)^n} - \ldots\ldots \text{to}$$

$\frac{1}{2} k+1$ terms or $\frac{1}{2}(k+1)$ terms according as k is even or odd.

Putting $x = \tan \theta$, we deduce

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \int_0^{\frac{\pi}{2}} \cos^{2n-2} \theta\, d\theta;$$

$$\int_0^\infty \frac{x^2\, dx}{(1+x^2)^n} = \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^{2n-4} \theta\, d\theta; \text{ etc.}$$

Thus each of these integrals may be expressed in terms of Gamma functions and B_k may be obtained.

To find a single* expression for the series which gives B_k, we write

$$\underline{|k}\, B_k=\int_0^{\frac{\pi}{2}}\left\{\cos^{2n-2}\theta-\frac{k_2}{2!}\sin^2\theta\cos^{2n-4}\theta+\frac{k_4}{4!}\sin^4\theta\cos^{2n-6}\theta-\ldots\right\}d\theta,$$

$$=\int_0^{\frac{\pi}{2}}\cos^{2n-k-2}\theta\left\{\cos^k\theta-\frac{k^2}{2!}\cos^{k-2}\theta\sin\theta+\ldots\right\}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\cos^{2n-k-2}\theta\cos k\,\theta\, d\theta$$

$$=\frac{\pi}{2^{2n-k-1}}\left\{\frac{\underline{|2n-k-2}}{\underline{|n-1}\,\underline{|n-k-1}}\right\},$$

as $2n-2$ is even (see Edward's *Integral Calculus*, Ex. 19, p. 106).

Hence $$B_k=\frac{\pi}{2^{2n-1}\,\underline{|n-1}}\left\{\frac{2^k\,\underline{|2n-k-2}}{\underline{|k}\,\underline{|n-k-1}}\right\}.$$

We thus get finally

$$\int_0^\infty\frac{\cos rx\,dx}{(1+x^2)^n}=\frac{\pi e^{-r}}{2^{2n-1}\,\underline{|n-1}}\sum_{k=0}^{k=n-1}\frac{2^k\,\underline{|2n-k-2}}{\underline{|k}\,\underline{|n-k-1}}r^k. \qquad \text{(B)}$$

§ 2. From $I=e^{-r}\{B_0+B_1 r+B_2\, r^2+\ldots\ldots B_{n-1}\, r^{n-1}\}$ integrating both sides with regard to r, we get

$$J\equiv\int_0^\infty\frac{\sin rx}{x(1+x^2)^n}dx=-e^{-r}\{B_0+B_1(1+r)+B_2(r^2+2r+2)+\ldots+$$
$$B_{n-1}[r^{n-1}+(n-1)_1 r^{n-2}+(n-1)_2 r^{n-3}+\ldots\ldots(n-1)_{n-1}]\}$$

To determine the constant we put $r=0$; then $(J)_0=0$; and the constant is equal to $B_0+B_1\,\underline{|1}+B_2\,\underline{|2}+\ldots+B_{n-1}\,\underline{|n-1}$.

Thus $$J=\{B_0+B_1\,\underline{|1}+B_2\,\underline{|2}+\ldots+B_{n-1}\,\underline{|n-1}\}$$
$$-e^{-r}\{B_0+B_0(1+r)+B_2(r^2+2r+2)+\ldots+$$
$$B_{n-1}\,[r^{n-1}+(n-1)_1 r^{n-2}+\ldots\ldots\underline{|n-1}]\}. \qquad \text{(C)}$$

§ 3. To find $\int_0^\infty\frac{\sin^2 rx\,dx}{(1+x^2)^n}\equiv K$ (say), we differentiate K with regard to r; we thus get

$$\frac{dK}{dr}=\int_0^\infty\frac{\sin 2rx}{x(1+x^2)n}=\{B_0+B_1\,\underline{|1}+B_2\,\underline{|2}+\ldots B_{n-1}\,\underline{|n-1}\}$$
$$-e^{-2r}\{B_0+B_1\;(2r+1)+B_2\;(4r^2+4r+2)+\ldots+B_{n-1}$$
$$[(2r)^{n-1}+(n-1)_1\,(2r)^{n-2}+(n-1)_2\,(2r)^{n-3}+\ldots+\underline{|n-1}]\}.$$

* This expression is due to Prof. K. J. Sanjana.

Integrating both sides with regard to r, and determining the constant as before, we get the value of K.

Now,

$$\int e^{-2r}\{(2r)^{n-1}+(n-1)_1(2r)^{n-2}+(n-1)_2(2r)^{n-3}+\ldots\underline{|n-1}\}\,dr$$

$$=-\frac{e^{-2r}}{2}[(2r)^{n-1}+(n-1)_1(2r)^{n-2}+(n-1)_2(2r)^{n-3}+\ldots+\underline{|n-1}$$

$$+(n-1)_1(2r)^{n-2}+(n-1)_2(2r)^{n-3}+\ldots+\underline{|n-1}$$

$$+(n-1)_2(2r)^{n-3}+\ldots+\underline{|n-1}$$

$$+\ldots\ldots\ldots\ldots$$

$$\text{etc.}]$$

$$=-\frac{e^{-2r}}{2}\{(2r)^{n-1}+2.(n-1)_1(2r)^{n-2}+3.(n-1)_2(2r)^{n-3}+\ldots+\underline{|n}\}.$$

The value of the constant is seen to be

$$-\tfrac{1}{2}\{B_0+B_1\underline{|2}+B_2\underline{|3}+\ldots+B_{n-1}\underline{|n}\};$$

thus we find

$$K=\int_0^\infty\frac{\sin^2 rx}{x^2(1+x^2)^n}dx=-\tfrac{1}{2}\{B_0+\underline{|2}\,B_1+\underline{|3}\,B_2+\ldots\underline{|n}\,B_{n-1}\}$$

$$+r\{B_0+\underline{|1}\,B_1+\underline{|2}\,B_2+\ldots\underline{|n}\,B_{n-1}\}$$

$$+\frac{e^{-2r}}{2}\{B_0+B_1(2r+2)+B_2(4r^2+8r+6)$$

$$+B_3(8r^3+24r^2+36r+24)+\ldots\ldots\ldots\ldots+B_{n-1}[(2r)^{n-1}+2.(n-1)_1(2r)^{n-2}+3.(n-1)_2(2r)^{n-3}+\ldots+\underline{|n}\}.\ \ldots\ldots\ldots\ldots(D).$$

§ 4. The following particular results may be noticed:

(i) Put $n=1$; then (B) becomes

$$\int_0^\infty\frac{\cos rx\,dx}{(1+x^2)}=\frac{\pi e^{-r}}{2}\qquad\ldots\qquad\ldots\ (1)$$

Making $r=1, 2$, etc.

$$\int_0^\infty\frac{\cos x}{1+x^2}dx=\frac{\pi}{2e},\ \int_0^\infty\frac{\cos 2x\,dx}{(1+x^2)}=\frac{\pi}{2e^2},\ \text{etc.}\ \ldots\qquad\ldots\ (2)$$

As $B_0=\frac{\pi}{2}$, the formula (C) becomes

$$\int_0^\infty\frac{\sin rx\,dx}{x(1+x^2)}=\frac{\pi}{2}(1-e^{-r});\qquad\ldots\qquad\ldots\ (3)$$

whence, making $r=1, 2$, etc.

$$\int_0^\infty\frac{\sin x}{x(1+x^2)}dx=\frac{\pi}{2}\left(\frac{e-1}{e}\right);$$

$$\int_0^\infty\frac{\sin 2x}{x(1+x^2)}dx=\frac{\pi}{2^2e}(e^2-1);\ \text{etc.}\qquad\ldots\qquad\ldots\ (4)$$

Formula (D) gives

$$\int_0^\infty \frac{\sin^2 rx\,dx}{x^2(1+x^2)} = -\frac{1}{2}\cdot\frac{\pi}{2}+\frac{\pi}{2}r+\frac{e^{-2r}}{2}\cdot\frac{\pi}{2}$$

$$=\frac{\pi}{2}\left(-1+e^{-2r}+2r\right)\;; \qquad \dots \qquad \dots \ (5)$$

[Ex. 497, Journal, I. M. S., December 1913.]

Taking $r=1$, 2, etc.

$$\int_0^\infty \frac{\sin^2 x\,dx}{x^2(1+x^2)} = \frac{\pi}{4}(1+e^{-2}) = \frac{\pi}{4e^2}(1+e^2)\;;$$

$$\int_0^\infty \frac{\sin^2 2x\,dx}{x^2(1+x^2)} = \frac{\pi}{4e^4}(1+3e^4)\;;\text{ etc.} \qquad \dots \qquad \dots \ (6)$$

(ii) Take $n=2$; (B) now gives

$$\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^2} = \frac{\pi e^{-r}}{2^3}(\underline{|2}+2r) = \frac{\pi e^{-r}}{4}(1+r) \quad \dots \qquad \dots \ (7)$$

Putting $r=1$, 2, etc.

$$\int_0^\infty \frac{\cos x\,dx}{(1+x^2)^2} = \frac{\pi}{2e},\ \int_0^\infty \frac{\cos 2x\,dx}{(1+x^2)^2} = \frac{3\pi}{4e^2} \qquad \dots \qquad \dots \ (8)$$

As now $B_0=\frac{\pi}{4}$ and $B_1=\frac{\pi}{4}$, we obtain from the formulae (C) and (D),

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx = \frac{\pi}{2} - e^{-r}\left(\frac{\pi}{2}+\frac{\pi}{4}r\right)$$

$$=\frac{\pi}{4}\left\{\,2-e^{-r}(r+2)\,\right\} \qquad \dots \qquad \dots \ (9)$$

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)^2}dx = \frac{\pi}{8}\left\{\,-3+4r+e^{-2r}(2r+3)\,\right\} \quad \dots \qquad \dots (10)$$

(iii) Take $n=3$; (B) gives

$$\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^3} = \frac{\pi e^{-r}}{2^5\underline{|2}}\left\{\frac{\underline{|4}}{\underline{|2}}+\frac{2\underline{|3}}{\underline{|1}\,\underline{|1}}r+\frac{2^2\underline{|2}}{\underline{|2}}r^2\right\}$$

$$=\frac{\pi}{16}e^{-r}(3+3r+r^2) \qquad \dots \qquad \dots (11)$$

[See solution of Ex. 472, December Journal, 1913.)

Here $B_0=\frac{3\pi}{16}$, $B_1=\frac{3\pi}{16}$, $B_2=\frac{\pi}{16}$, so that from (C) and (D) we obtain

$$\int_0^\infty \frac{\sin rx\,dx}{x(1+x^2)^3} = \frac{\pi}{16}\left\{8-e^{-r}(8+5r+r^2)\right\} \qquad \dots \qquad \dots (12)$$

[See Ex. 531, February Journal, 1914.]

Also $$\int_0^\infty \frac{\sin^2 rx\,dx}{x^2(1+x^2)^3} = \frac{\pi}{32}\left\{\,-15+16r+e^{-2r}(15+14r+4r^2)\,\right\} \qquad (13)$$

(iv) Similarly, we easily obtain :—

$$\int_0^\infty \frac{\cos rx\, dx}{(1+x^2)^4} = \frac{\pi e^{-r}}{96}(15+15r+6r^2+r^3) \quad \ldots \quad \ldots(14)$$

$$\int_0^\infty \frac{\sin rx\, dx}{x(1+x^2)^4} = \frac{\pi}{96}\{48-e^{-r}(48+33r+9r^2+r^3)\} \quad \ldots(15)$$

$$\int_0^\infty \frac{\sin^2 rx\, dx}{x^2(1+x^2)^4} = \frac{\pi}{192}\{-105+96r+e^{-r}(105+114r+48r^2+8r^3)\} \quad (16)$$

$$\int_0^\infty \frac{\cos rx\, dx}{(1+x^2)^5} = \frac{\pi}{768}e^{-r}(105+105r+45r^2+10r^3+r^4) \quad \ldots(17)$$

$$\int_0^\infty \frac{\sin rx\, dx}{x(1+x^2)^5} = \frac{\pi}{768}\{384-e^{-r}(384+279r+87r^2+14r^3+r^4)\} \quad (18)$$

20th March 1914. T. P. TRIVEDI.

The Face of the Sky for July and August 1914.

Sidereal time at 8 p.m.

	July.				August.		
D.	H.	M.	S.		H.	M.	S.
1	6	58	21	...	9	0	35
8	7	25	57	...	9	28	11
15	7	53	33	...	9	55	47
22	8	21	9	...	10	23	23
29	8	48	45	...	10	50	59

Phases of the Moon.

	July.				August.		
	D.	H.	M.		D.	H.	M.
Full Moon ...	7	7	30 P.M.	...	6	6	11 A.M.
Last Quarter ...	15	1	2 „	...	14	6	26 „
New Moon ...	23	8	8 A.M.	...	21	5	56 P.M.
First Quarter ...	30	5	21 „	...	28	10	22 A.M.

Eclipses.

On August 21, there is a partial eclipse of the Sun, partly visible at Madras.

First contact	...	6·5 P.M.
Middle „	...	6·58 „
Last „	...	8·1 „

The Planets.

Mercury is stationary on July 2 when it begins to retrograde. It is in inferior conjunction with the Sun on July 16. It is stationary on July 27 and attains its greatest elongation (19°14′ W) on August 5. It is in superior conjunction with the Sun on August 30. It is in conjunction with the Moon on July 22 and on August 20 and with Neptune on August 10.

Venus is an evening star. It is in conjunction with the Moon on July 26 and August 25 and with Mars on August 6.

Mars, which is approaching conjunction with the Sun, is in conjunction with the Moon on August 24 at 5·56 P.M.

Jupiter is in opposition to the Sun on August 11. It is in conjunction with the Moon on July 10.

Saturn, which is in conjunction with the Sun in the middle of June, is in conjunction with the Moon on July 20 at 8·13 and on August 17.

Uranus is in opposition to the Sun on August 3. It is in conjunction with the Moon on July 9.

Neptune is in conjunction with the Sun on July 21 and with the Moon on July 23 and August 19.

V. RAMESAM.

The Late Mr. W. Gallatly.

The *Educational Times*, dated March 2, announces the sad news of the sudden death at Boscombe, from heart failure, of Mr. William Gallatly, M.A. His name must be familiar to readers of our Journal as a valuable contributor in Modern Geometry. " A scholar of Pembroke College, Mr. Gallatly obtained his University training and his M.A. Degree at the University of Cambridge. He was both an author and a compiler of works on mathematical subjects, and last year saw the publication of the second (enlarged and carefully revised) edition of his *Modern Geometry of the Triangle.*"

SOLUTIONS.

Question 486.

(A. N. RAGHAVACHAR, M.A.) :—If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4+px^3+qx^2+rx+s=0$, form the equation whose roots are

$$\frac{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)}{\beta+\gamma-\alpha-\delta}, \text{ \&c., \&c.}$$

Additional Solution by D. Krishnamurti.

Let $ax^4+4\,bx^3+6cx^2+4\,dx+e=0$ be the proposed equation. Then taking Ferrari's solution of the biquadratic (Burnside and Panton *Theory of Equations*, Vol. I, § 63), we see that

$$\beta+\gamma=-\frac{2}{a}(b-\text{M}),\ \beta\gamma=\frac{c+2a\theta-\text{N}}{a},$$

$$\alpha+\delta=-\frac{2}{a}(b+\text{M}),\ \alpha\delta=\frac{c+2a\theta+\text{N}}{a}.$$

If ρ be a root of the required equation, we find

$$\rho=\frac{b\text{N}-\text{M}\,(c+2a\theta}{a\text{M}}$$

$$\therefore \qquad a\rho+c+2a\theta=\frac{b\text{N}^2}{\text{MN}}$$

$$=\frac{b(c+2a\theta)^2-abe}{bc+ad+2ab\theta},$$

whence, after simplification,

$$\rho=\frac{cd-be+2ad\theta}{bc-ad+2ab\theta}.$$

The required equation is obtained from the reducing cubic by the above homographic transformation. The actual equation is seen to be with the usual notation

$$[cd-be-\rho(bc-ad)]^3$$
$$-\text{I}[cd-be-\rho(bc-ad)](b\rho-d)^2+2\text{J}(b\rho-d)^3=0.$$

Question 502.

(T. P. TRIVEDI, M.A., L.L.B.) :—Prove that the equation of the sides of a regular polygon of $(2n+1)$ sides may be written in the form

$$\text{P}\sin(2n+1)\alpha+(-1)^n\text{Q}\cos(2n+1)\alpha-\text{R}=0,$$

where

$$\text{P}\equiv y^{2n+1}-\frac{2n+1)_2}{2\,!}y^{2n-1}.x^2+\frac{(2n+1)_4}{4\,!}y^{2n-3}.x^4-.(-1.)^n(2n+1)y.x^{2n},$$

$$Q \equiv x^{2n+1} - \frac{(2n+1)_2}{2\,!} x^{2n-1} y^2 + \frac{(2n+1)_4}{4\,!} x^{2n-3} y^4 - \ldots\ldots (-1^n)(^{2n}+1) x y^{2n},$$

$$R \equiv n a r^{2n} - \frac{n(n^2-1^2)}{3\,!} a^3 r^{2n-2} + \frac{n(n^2-1^2)(n^2-3^2)}{5\,!} a^5 r^{2n-4} \ldots \quad \text{to } (n+1)$$

terms. [Extension of Question 479.]

Solution by N.B. Pendse.

Let the centre of the polygon be the origin and the perpendicular on one side make an angle α with the initial line; and let a be the radius of the inscribed circle. Then

$$r \cos\left(\alpha + \frac{\pi k}{2n+1} - \theta\right) = a,$$

is the equation representing a side of the polygon for different values of k.

Put $\left(\alpha + \frac{2\pi k}{2n+1} - \theta\right) = \phi$, and we have

$$r \cos \phi = a,$$

and $$\cos (2n+1)\phi = \cos \{ (2n+1)(\alpha - \theta) \}.$$

Expanding $\cos (2n+1)\theta$, $\sin (2n+1)\theta$ in terms of $\cos\theta$, $\sin\theta$ and putting x/r, y/r in place of $\cos\theta$, $\sin\theta$, we find

$$r^{2n+1}. \cos (2n+1)\phi = Q \cos (2n+1)\alpha + (-1^n).P \sin (2n+1)\alpha.$$

Again, expand $\cos(2n+1)\phi$ in terms of $\cos\phi$ and substitute a/r for the latter. We have

$$r^{2n+1} \cos (2n+1)\phi = (-1)^n R.$$

Hence, multiplying by $(-1)^n$ throughout,

$$R = (-1)^n Q \cos (2n+1)\alpha + P \sin (2n+1)\alpha,$$

which is the result stated in the question

Question 503.

(R. N. Apte, M.A., L.L.B.) :—A circle is drawn within an ellipse concentric with the ellipse. A tangent to this circle cuts the ellipse in P,P′. The tangents to the ellipse $(x^2/a^2 + y^2/b^2 = 1)$ at P, P′ meet in T, the normals meet in N. If p is the perpendicular from the centre on TN ; θ is the angle between PP′ and TN; $a^2+\lambda$, $a^2+\mu$ are the squares of the transverse semi-axes of the confocal conics through T ; prove that for the different positions of PP′

$$\frac{1}{\lambda} + \frac{1}{\mu} \propto p. \sec \theta.$$

Solution by N. B. Pendse.

Let P, P′ be the points α, β. Then obviously T is

$$\{ a\cos\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta),\ b\sin\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta) \}.$$

i.e., (al, bm) say.

Also N is

$$\{ lc^2\cos\alpha\cos\beta/a, -mc^2\sin\alpha\sin\beta/b \}.$$

The equation of TN is therefore

$$\frac{x-al}{y-bm}=\frac{b\cos\frac{1}{2}(\alpha+\beta)(a^2-c^2\cos\alpha\cos\beta)}{a\sin\frac{1}{2}(\alpha+\beta)\,|\,(b^2+c^2\sin\alpha\sin\beta)}.$$

If d is the distance from the origin to the point of intersection of TN and the perpendicular on PP′, we have $p.\sec\theta=d$;

and

$$\frac{y}{x}=\frac{a\sin\frac{1}{2}(\alpha+\beta)}{b\cos\frac{1}{2}(\alpha+\beta)}.$$

$$ad\sin\tfrac{1}{2}(\alpha+\beta)=y.\{ a^2\sin^2\tfrac{1}{2}(\alpha+\beta)+b^2\cos^2\tfrac{1}{2}(\alpha+\beta) \}^{\frac{1}{2}}$$
$$=y.n \text{ say},$$
$$bd\cos\tfrac{1}{2}(\alpha+\beta)=xn.$$

Hence substituting in TN

$$d=\frac{c^2(b^2\cos\alpha\cos\beta+a^2\sin\alpha\sin\beta)n}{ab\{ b^2-a^2+(a^2-b^2)(\cos\alpha\cos\beta+\sin\alpha\sin\beta) \}\cos\frac{1}{2}(\alpha-\beta)}$$
$$=-\frac{n(b^2\cos\alpha\cos\beta+a^2\sin\alpha\sin\beta)}{ab\{ 1-\cos(\alpha-\beta) \}\cos\frac{1}{2}(\alpha-\beta)}.$$

Now λ, μ are the roots of

$$\frac{a^2\cos^2\frac{1}{2}(\alpha+\beta)}{a^2+k}+\frac{b^2\sin^2\frac{1}{2}(\alpha+\beta)}{b^2+k}=\cos^2\tfrac{1}{2}(\alpha+\beta),$$

and the sum of the reciprocals of the roots is therefore

$$\frac{a^2\cos^2\frac{1}{2}(\alpha-\beta)-a^2\cos^2\frac{1}{2}(\alpha+\beta)+b^2\cos^2\frac{1}{2}(\alpha-\beta)-b^2\sin^2\frac{1}{2}(\alpha+\beta)}{a^2b^2\sin^2\frac{1}{2}(\alpha-\beta)}$$
$$=\frac{a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta}{a^2b^2\sin^2\frac{1}{2}(\alpha-\beta)}.$$

$$\therefore\quad \frac{1}{\lambda}+\frac{1}{\mu}=-\frac{2d\cos\frac{1}{2}(\alpha-\beta)}{abn}.$$

The condition that PP′ should touch a concentric circle of given radius reduces to $\cos\frac{1}{2}(\alpha-\beta)\propto n$.

Hence

$$\frac{1}{\lambda}+\frac{1}{\mu}\propto d.$$

Question 509.

(K. APPUKUTTAN ERADY, M. A.) :—Forces act at the point (f,g,h) within the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, and are represented by the normals from the point to the surface. Show that the resultant acts along the line whose direction cosines are

$$\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}+1\right)f, \text{ \&c, \&c,}$$

and find its magnitude.

Solution by R. Srinivasan, M. A., and M. K. Veeraraghavan.

If the normal at (x, y, z) passes through $(f,g,h,)$, we have

$$\frac{f-x}{x/a^2}=\frac{g-y}{y/b^2}=\frac{h-z}{z/c^2}.$$

$$\therefore \qquad y=\frac{b^2gx}{(b^2-a^2)x+a^2f} \quad \text{and } z=\frac{c^2hx}{(c^2-a^2)x+a^2f}$$

Since (x, y, z) is on the ellipsoid,

$$\frac{x^2}{a^2}+\frac{b^2g^2x^2}{\{(b^2-a^2)x+a^2f\}^2}+\frac{c^2h^2x^2}{\{(c^2-a^2)x+a^2f\}^2}=1.$$

Simplifying, we get

$$x^6(b^2-a^2)^2(c^2-a^2)^2+2x^5\{a^2f(c^2-a^2)(b^2-a^2)^2$$
$$+a^2f(b^2-a^2)(c^2-a^2)^2\}+\dots\dots\dots=0.$$

$\therefore$ If $x_1x_2\dots\dots x_6$ be the roots,

$$\Sigma x=2a^2f\left(\frac{1}{a^2-b^2}+\frac{1}{a^2-c^2}\right)$$
$$=2f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}+2\right).$$

Now, if (α,β,γ) be the mean centre of the feet of the six normals, the direction cosines of the resultant will be proportional to

$$\alpha-f,\ \beta-g,\ \gamma-h\,;$$

i.e., to $\quad \frac{1}{6}\Sigma x-f,\ \frac{1}{6}\Sigma y-g,\ \frac{1}{6}\Sigma z-h\,;$

i.e., to $\quad \frac{1}{3}f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}+2\right)-f$, &c., &c.

i.e., to $\quad f\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}-1\right)$, &c., &c.

The magnitude R of the resultant is given by

$$R^2=(f-\alpha)^2+\dots\dots\dots$$
$$=\frac{1}{9}\left\{f^2\left(\frac{b^2}{a^2-b^2}+\frac{c^2}{a^2-c^2}-1\right)^2+\dots\dots\dots\right\}.$$

Question 513.

(Professors, T. P. TRIVEDI & K. J. SANJANA) :—Find the sum of the series

$$1-\frac{h}{h-k+2}+\frac{h(h-1)}{(h-k+2)(h-k+3)}-\frac{h(h-1)(h-2)}{(h-k+2)(h-k+3)(h-k+4)}+\ldots\ldots$$

where h is a positive integer and k any rational number.

Solution by R. Srinivasan, M.A., and K. Rangachari.

Consider the series

$$\frac{1}{m+1}-\frac{n-1}{(m+1)(m+2)}+\frac{(n-1)(n-2)}{(m+1)(m+2)(m+3)}-\ldots\ldots$$

This can be easily shewn to be equal to $\frac{1}{m+n}$ (*Vide* : Smith's *Algebra*, page 431, Example 14.)

If in this we put $h-k+1$ for m and h for n, we get

$$\frac{1}{h-k+2}-\frac{h-1}{(h-k+2)(h-k+3)}\ldots\ldots\ldots+=\frac{1}{2h-k+1}.$$

The given series is thus equal to

$$1-h.\frac{1}{2h-k+1}=\frac{h-k+1}{2h-k+1}.$$

Question 515.

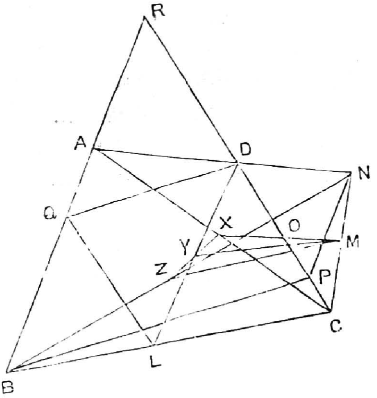
(A. C. L. WILKINSON M.A., F.R.A.S.) :—ABCD is a quadrilateral. DQ, BP are any two parallel straight lines meeting AB, CD respectively in Q and P; QL parallel to CD meets BC in L; and PN parallel to AB meets AD in N. Prove that the middle points of AC, DL, BN are collinear.

Solution by K. Rangachari.

Let BA, CD produced meet in R and let X, Y, Z, M and O be the middle points of AC, DL, BN, CN and CD respectively. Join XM, XY, XZ OY and MZ.

In the ΔACN, X and M are the mid-points of CA and CN. Hence (M passes through O and it is parallel to and a half of AN. Similarly

XO is a half of AD, YO is parallel to and a half of LC, and ZM is parallel to and a half of BC.



In the triangle BRC, LQ is parallel to CR. Hence CL : CB = RQ : RB.

In the triangles RAD, PDN, PN is parallel to AR. Hence PD : DR = ND : DA.

Thus PR : RD = NA : AD = RB : RQ, since in the $\triangle$PRB, QD is parallel to BP.

$$\therefore \quad AN : AD = BC : CL$$

$$\therefore \quad XM : XO = ZM : YO.$$

In the $\triangle$s XOY and XMZ, $X\hat{O}Y = X\hat{M}Z$ and the ratios of the sides about these angles are equal. Therefore $X\hat{Y}O = X\hat{Z}M$.

Hence X, Y, Z are collinear.

Question 516.

(K. J. Sanjana M.A.) :—TP, TQ tangents to a conic of centre C and focus F, cut the auxliary circle in YZ; and FW is perpendicular to the chord of contact; a tangent of the conic perpendicular to TC cuts FY, FZ, FW in y, z, w, respectively. Prove that $yw = wz$ and ennunciate the corresponding theorem for the parabola.

Solution by D. Krishnamurti and K. Rangachari.

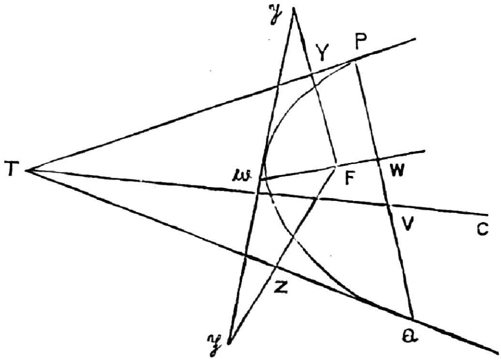
Since Y and Z are points of intersection of the auxiliary circle with the tangents, FY, FZ are $\perp^r$ to TP, TQ. Let CT cut PQ in V.

Comparing Δs Fwz and QVT, we see that they are similar, since Fw, wz, zF are respectively $\perp^r$ to QV, VT and TQ.

$\therefore$ $wz : TV = Fw : QV$ ... ... (1)

In the same way we can show that Δ^s F wy and PVT are similar.

$\therefore$ $wy : TV = Fw : PV$ ... ... (2)



From (1) and (2), since PV = VQ,

$$wz : TU = wy : TU$$

$$\therefore \quad wz = wy.$$

[*N.B.*—The proposition will hold even if the line ywz is not a tangent, provided it is $\perp^r$ to CT.]

In the case of a parabola the proposition will be :—

The portion of the tangent at the vertex of a parabola, intercepted by any two tangents, is bisected by the perpendicular from the focus on the chord of contact.

Question 519.

(V. Anantaraman):—Given the base and the vertical angle of a triangle, find the locus of the centre of the circle passing through the three excentres.

Solution by R. Srinivasan, M.A., D. Krishnamurti and K. Rangachari.

We know the circumcircle and the locus of the incentre I (say the circle X on BC). If $O,_1O'$ be the circumcentres of the triangles ABC, and $I_1I_2I_3$, we know $OO'=IO$; and O is fixed.

∴ O′ divides OI externally in the ratio 1 : 2.

Hence the locus of O′ is a curve similar to that of I, and is a circlee

It is easily seen that this circle equals the locus of the incentre and has its centre at the highest point of the circumcircle.

Question 522.

(V. V. S Narayan) :—Given a parallelogram drawn on paper and an ungraduated straight edge, show how to trisect a given finit. straight line on the same paper.

Solution by M. K. Veeraraghavan.

Through the extremities of the given straight line PQ, draw parallels to the sides of the given parallelogram with the ruler.

[For construction see Askwith's *Pure Geometry*, art. 63].

Let PRQS be the parallelogram so formed with PQ for a diagonal. Join RS cutting PQ at C. Through C draw parallels to the sides of the parallelogram, meeting SP, SQ at A and B respectively. Then RA and RB trisect PQ.

The proof is almost obvious.

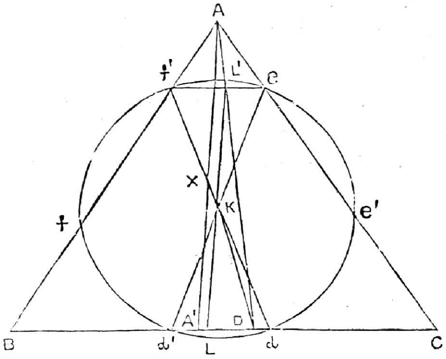
Question 529.

(N. P. Pandya) :—In a triangle ABC, D,E,F are the middle points of BC, CA, AB. If AX, BY, CZ are perpendiculars on the sides of DEF' show geometrically that DX, EY FZ are concurrent at the symmedian point of ABC.

Solution by R. Srinivasan M. A.

Draw the cosine circle cutting BC, CA, AB in d', d ; e', e ; f', f ; then K the symmedian point is its centre. Draw LKL′ ⊥r to BC

meeting BC in L and, ef' in L′. Then L′ is the mid-point of ef' which is =to BC.



∴ AL′ passes through D.

If AA′ be the altitude, X is the mid-point of AA′ and K is the mid-point of LL′.

∴ DKX is a line, since DL′A is a line.

Similarly K lies on EY and FZ.

Question 532.

(R. Vythynathaswamy) —If 2A, 2B, 2C are the mutual inclinations of any three lines in space and 2L, 2M, 2N the angles which any fourth straight line makes with them, show that the following identical relation holds :

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 4 \\ 1, & 0, & \sin^2 C, & \sin^2 B, & \sin^2 L, & 1 \\ 1, & \sin^2 C, & 0, & \sin^2 A, & \sin^2 M, & 1 \\ 1, & \sin^2 B, & \sin^2 A, & 0, & \sin^2 N, & 1 \\ 1, & \sin^2 L, & \sin^2 M, & \sin^2 N, & 0, & 1 \\ 4, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Solution by R. J. Pocock, B.A., B.Sc., F.R.AS.

Take the three given lines as is coordinate axes and let l, m, n be the ratios of the fourth line. Then

$$\cos 2L = l + m\cos 2C + n\cos 2B$$
$$\cos 2M = l\cos 2C + m + n\cos 2A$$
$$\cos 2N = l\cos 2B + m\cos 2A + n.$$

Multiply by l, m, n and add : $l\cos 2L + m\cos 2M + n\cos 2N = 1$.

We have $\quad 2(l+m+n-1) - 2l - 2m - 2n + 2 = 0$

$$l+m+n-1 \qquad -2m\sin^2 C - 2n\sin^2 B + 2\sin^2 L = 0.$$
$$l+m+n-1-2l\sin^2 C \qquad -2n\sin^2 A + 2\sin^2 M = 0.$$
$$l+m+n-1-2l\sin^2 B - 2m\sin^2 A \qquad +2\sin^2 N = 0.$$
$$l+m+n-1-2l\sin^2 L - 2m\sin^2 M - 2n\sin^2 N \qquad = 0.$$

From these
$$\begin{vmatrix} 2, & 1, & 1, & 1, & 1, \\ 1, & 0, & \sin^2 C, & \sin^2 B, & \sin^2 L \\ 1, & \sin^2 C, & 0, & \sin^2 A, & \sin^2 M \\ 1, & \sin^2 B, & \sin^2 A, & 0, & \sin^2 N \\ 1, & \sin^2 L, & \sin^2 M, & \sin^2 N, & 0 \end{vmatrix} = 0.$$

Multiplying the 1st row by 2 and bordering the determinant, we obtain

$$\begin{vmatrix} 4, & 2, & 2, & 2, & 2, & 0 \\ 1, & 0, & \sin^2 C, & \sin^2 B, & \sin^2 L, & 0 \\ 1, & \sin^2 C, & 0, & \sin^2 A, & \sin^2 M, & 0 \\ 1, & \sin^2 B, & \sin^2 A, & 0, & \sin^2 N, & 0 \\ 1, & \sin^2 L, & \sin^2 M, & \sin^2 N, & 0, & 0 \\ 4, & 1, & 1, & 1, & 1, & 4 \end{vmatrix} = 0.$$

Subtracting the 1st column from the sixth and then the sixth row from the first row, we get the required result.

QUESTIONS FOR SOLUTION.

548. (A. NARASINGA RAO) :—If $u_r = \frac{x^r}{1-x^r}$, shew that the infinite series $u_1+u_2+u_3+u_5-u_6+u_7-u_{10}\ldots = x\frac{(1+x)}{(1-x)}$, where the suffixes of u are the numbers having non-repeated prime factors, the terms being taken with a + ve or — ve sign according as the number of such factors is odd or even.

549. (K. J. SANJANA, M.A.) :—If $\sum_n^r \equiv 1^r - 2^r + 3^r \ldots + (-1)^{n+1} n^r$,

prove $\sum_n^r + \binom{r+1}{1}\sum_{n-1}^r + \binom{r+2}{2}\sum_{n-2}^r + \ldots\ldots = 0$, if r is even and n is an integer equal to or greater than r. Show how the value of the series may be found when r is odd.

550. (K. J. SANJANA) :—At a game of bridge recently played it was found that every player had three cards of each of three suits and four cards of the remaining suit: what is the chance of this distribution of the cards ?

551. (T. P. TRIVEDI, M.A., LL.B.) :—Prove that

$$\int\int\int (abcdfghlmn)(xyz_1)^2 \, dx \, dy \, dz,$$

taken throughout a tetrahedron bounded by the co-ordinate planes and the plane $x/p + y/q + z/r = 1$ is equal to

$$\frac{pqr}{60}\{(ap^2+bq^2+cr^2+fqr+gpr+hpq)+5(lp+mq+nr)+10d\}.$$

532. (S. KRISHNASWAMI AIYANGAR) :—Find the sum of

$$\frac{2}{3}x + \frac{1}{3\cdot5\cdot2}\frac{x^2}{1!} + \frac{1\cdot3}{5\cdot7\cdot2^3}\frac{z^3}{3!} + \frac{1\cdot3\cdot5}{7\cdot9\cdot2^5}\frac{x^4}{3!} + \ldots\ldots$$

$$+ \frac{1\cdot3\cdot5\ldots\ldots2n-1}{(2n+1)(2n+3)2^{2n-1}}\cdot\frac{x^{n+1}}{n!} + \ldots$$

553. (S. KRISHNASWAMI AIYANGAR) :—A B C is a triangle such that its sides subtend equal angles at one focus S of an incribed conic. With S as focus four more conics are drawn circumscribing the triangle. Show that—

(i) The points of contact with the sides of the inscribed conic are on the focal radii through the vertices of the triangle.

(ii) The directrix of one of the circum-conics is identical with the directrix of the inscribed conic corresponding to S.

(iii) The directrix of one of the other circum-conics and its tangent through C meet at the point of intersection of AB and the directrix of the in-conic.

(iv) If this point is P_3, then P_3 S C′ is a right angle where C′ is the point of contact of the inscribed conic with the side AB.

554. (T. P. Trivedi, M.A., LL.B.) :—If V denote the entire volume of the solid

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n+\left(\frac{z}{c}\right)^n=1,$$

and A the whole area of its trace on the xy-plane, prove that

$$\frac{V}{A}=\frac{4c}{3n}.\quad B\left(\frac{2}{n},\frac{1}{n}\right),$$

and deduce that for the solid

$$\left(\frac{x}{a}\right)^{\frac{2p}{2p+1}}+\left(\frac{y}{b}\right)^{\frac{2p}{2p+1}}+\left(\frac{z}{c}\right)^{\frac{2p}{2p+1}}=1,$$

$$\frac{V}{A}=\frac{8}{9}\,\frac{c(p+1)(2p+1)}{p(2p+3)(4p+3)}.$$ (Extension of Question 523.)

555 (R. Vythynathaswamy) :—P is a point on one of a system of curves $f\ (r_1,\ r_2,\ \alpha)=0$ where α is a variable parameter. If dn is the distance at the point P between this curve and its consecutive, whose parameter is $\alpha+d\alpha$, shew that

$$\frac{dn}{d\alpha}=\frac{1}{PQ}.\frac{\partial f}{\partial \alpha}$$

PQ being the diagonal of a parallelogram whose adjacent sides are along r_1, r_2 and equal to $\frac{\partial f}{\partial r_1}, \frac{\partial f}{\partial r_2}$ respectively.

556. (R. Vythynathaswamy) :—Shew that in a cissoid the product of the perpendicular from the cusp on a tangent and the length of the tangent between point of contact and the asymptote varies as the ordinate of the point of contact.

557. (Professor Sanjana) :—At a point P of a parabola the normal chord PQ is drawn meeting the axis in G and PN is the ordinate to the axis, prove that PQ. AN $=2$ SP. PG.

If from P two chords PP_1, PP_2 are drawn normal to the curve at P_1, P_2, prove further that

$$SP_1 \cdot SP_2 = AS \cdot SP, \quad PP_1 \cdot PP_2 = PQ \cdot AN,$$

where S is the focus.

Shew also that the circle $P_1 P_2 S$ passes through the circumcentre of PP_1P_2.

558. (N. P. Pandya) :—Solve completely

$$y+(9y+x)\frac{dy}{dx}+3x\left(\frac{dy}{dx}\right)^2+x\left(3y+3x\frac{dy}{dx}+x\right)\frac{d^2y}{dx^2}+x^2y\frac{d^3y}{dx^3}=0.$$

559. (N. Sankara Aiyar, M.A.) :—Evaluate

$$\int_0^\infty y^{n-1}\coth y\,dy,$$

and
$$\int_0^\infty y^{n-1}\operatorname{cosech}^2 y\,dy.$$

(Suggested by Question 465 of Mr. S. Narayana Aiyar, M.A.)

560. (N. Sankara Aiyar, M.A.) :—Find the volume contained between the two ellipsoids $x^2/a^2+y^2/b^2+z^2/c^2=1$, and $x^2/b^2+z^2/c^2+y^2/a^2=1$.

561. (P. A. Subramania Iyer, B.A., L.T.) :—Solve completely

$$(1+3x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-4y=0.$$